DISCONTINUOUS BOUNDARY CONDITIONS AND THE DIRICHLET PROBLEM*

BY

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It has been suggested that in the Dirichlet problem there is something essentially antagonistic between the utmost degree of generality attainable as regards the geometrical character of the boundary and the utmost of generality attainable as regards the boundary values assumed over this boundary. The purpose of this paper is to show that the solution of the Dirichlet problem merely for continuous boundary conditions at once entrains the unique determination of a harmonic function correlated with discontinuous boundary conditions of a very general sort. The logical tool employed to this end is the Daniell integral.† In the stress laid on generalized types of integration this paper is closely akin to one of G. C. Evans,‡ but it appears to the author that though the theory of Evans gives more detailed information concerning the character of the solutions of the Dirichlet problem in the neighborhood of the boundary, it is less direct than the present theory, and less extensible to regions of infinite connectivity or higher dimensionality.

I. THE POTENTIAL AT A POINT AS A LINEAR FUNCTIONAL OF THE BOUNDARY CONDITIONS

Let R be any open set of points in n-space connected in the sense that any two of its points form extremities of a polygonal line lying entirely within it, and not extending to infinity. Let the Dirichlet problem be solvable over R for any continuous boundary conditions on C. That is, if $U(P)$ is defined for every point P on C, and if

$$\lim_{\overline{PQ} \to 0} U(P) = U(Q)$$

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* Presented to the Society, April 28, 1923.

† *A general form of integral,* Annals of Mathematics, ser. 2, vol. 19 (1918), p. 279.

‡ *Problems of potential theory,* Proceedings of the National Academy of Sciences, vol. 7 (1921), pp. 89-98.

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for every Q on C, then there is a function $u(P)$, defined and continuous over $R+C$, harmonic over R, and reducing to U on C. With the aid of a form of generalized integral due to Daniell, we shall associate with any discontinuous function $V(P)$ of a very wide class of functions defined over C a unique function $v(P)$ defined and harmonic over R, which we might naturally call the solution of the boundary value problem corresponding to $V(P)$.

Let Q be a given point in the interior of R. Then $u(Q)$ may be regarded as a functional of $U(P)$. To symbolize this point of view, let us write

$$I_Q(U) = u(Q).$$

Since the functions harmonic over R form a linear set, we have

(C) $$I_Q(CU) = CI_Q(U),$$

and

(A) $$I_Q(U_1 + U_2) = I_Q(U_1) + I_Q(U_2).$$

Moreover, because of the fact that no function harmonic over R has extrema over R,

(P) $$I_Q(U) \geq 0 \quad \text{if} \quad U \geq 0.$$

Another property of I_Q is that
(L) If $U_1, U_2, \ldots U_n, \ldots$ is a sequence of continuous functions defined over C, and

$$U_1 \geq U_2 \geq \cdots \geq 0 = \lim U_n$$

for every point on C, then

$$\lim I_Q(U_n) = 0.$$

To prove this, it is sufficient to show that

(1) $$\lim \max U_n = 0,$$

since

$$I_Q(U_n) \leq \max U_n.$$

If (1) were false, it would be possible to find a positive number ε and a sequence $\{P_n\}$ of points on C such that for every n

$$U_n(P_n) > \varepsilon.$$

This sequence will have at least one limit element N, which will belong to C, as C is a closed, bounded set. By hypothesis

$$U_1(N) \geqq U_2(N) \geqq \cdots \geqq 0 = \lim U_n(N).$$

Hence there is a value of n, say ν, such that

(2) $$U_\nu(N) < \varepsilon/2.$$

Let $\{N_n\}$ be a sequence selected from the P_n's and approaching N as a limit. It follows from the hypothesis of (L) that for every N_n from some stage $n = \mu$ on

$$U_\nu(N_n) > \varepsilon.$$

Hence, since U_ν is continuous,

(3) $$U_\nu(N) \geqq \varepsilon.$$

Since formulas (2) and (3) contradict one another, (1) and hence (L) is proved

The continuous functions defined over C form a linear set: that is, the sum of any two or a constant multiple of any one is also continuous. The absolute value of a continuous function is continuous. A continuous function is bounded over C. These properties of continuous functions are those attributed to a class T_0 by Daniell in the paper to which we have already referred. Moreover (C), (A), (P), and (L) show that the operator I_Q fulfils the conditions which he has laid down for an operator I on T_0. We are hence in a position to employ those definitions and theorems by which he enlarges the scope of the operator I.

II. The Daniell Extension of I_Q

In accordance with a definition of Daniell, a function U is of the class T_1 if $U_1 \leqq U_2 \leqq \cdots$ is a non-decreasing sequence of functions from T_0 and $U = \lim U_n$. Under these circumstances, by a theorem of Daniell,

$$I_Q(U_1) \leqq I_Q(U_2) \leqq \cdots,$$

and either lim $I_Q(U_n)$ exists or else $I_Q(U_n)$ becomes infinite. We shall write

$$(4) \qquad u(Q) = I_Q(U) = \lim I_Q(U_n) = \lim u_n(Q).$$

In general, we shall preserve the usage here indicated for the correlation of corresponding small and capital letters, applying the latter to functions on C, the former to the functions thereby determined as generalized integrals on R.

Now $u_n(Q)$ is harmonic over R. Hence the sequence $\{u_n\}$ is a monotone sequence of harmonic functions. It follows from a theorem of Harnack* that if $u(Q)$ is finite for any Q in R, it is finite and harmonic for every Q in R, while $u_n(Q)$ converges uniformly to $u(Q)$ over any closed region S interior to R.

In accordance with Daniell's definition, if V is any function defined on C, we shall write $\bar{I}_Q(V)$ for the lower bound of $I_Q(U)$ for all the functions U of class T_1 such that $U \geqq V$. Similarly, we make the definition

$$\underline{I}_Q(V) = -\bar{I}_Q(-V).$$

When we have $\bar{I}_Q(V) = \underline{I}_Q(V)$, we say that V is summable, and with Daniell write what is in our notation

$$v(Q) = I_Q(V) = \bar{I}_Q(V) = \underline{I}_Q(V).$$

When V is summable, it is clearly possible to find, whatever positive number ε may be, a pair of functions U_1 and U_2, such that U_1 and $-U_2$ belong to T_1, while

$$U_1 \geqq V \geqq U_2,$$

and

$$I_Q(U_1) + I_Q(-U_2) = u_1(Q) - u_2(Q) < \varepsilon.$$

The function $u_1(P) - u_2(P)$ is clearly harmonic. We now proceed as in the proof of Harnack's theorem.† Let us describe about Q any circle lying entirely within R. Let $u_1 - u_2$ assume the values W on the periphery of this circle, the radius of which we take to be a. Then over the interior of the circle

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* Osgood, *Lehrbuch der Funktionentheorie*, Leipzig, 1907, p. 615. This theorem is there proved for the two-dimensional case, but the proof may be extended at once to n dimensions. The restriction of the theorem to "Bereiche S" is not essential, as it applies at once to all connected open regions.

† The proof is here given in the form appropriate to a space of two dimensions, but *mutatis mutandis* is valid in n dimensions. For the sake of explicitness the language of two dimensions is used in several subsequent passages.

$$u_1 - u_2 = \frac{1}{2\pi} \int_0^{2\pi} W \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \psi) + r^2} \, d\psi.$$

Since

$$0 \leqq \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \psi) + r^2} \leqq \frac{a + r}{a - r} \qquad (r < a)$$

we have

$$u_1 - u_2 \leqq \frac{1}{2\pi} \int_0^{2\pi} W \frac{a + r}{a - r} \, d\psi$$

(5)
$$= \{u_1(Q) - u_2(Q)\} \frac{a + r}{a - r}$$

$$< \epsilon \frac{a + r}{a - r}.$$

It follows that within any circle about Q of radius less than a, by making ϵ approach 0, we shall make $u_1 - u_2$ approach 0 uniformly. Since $u_1(P) \geqq \underline{I}_P(V)$ and $u_2(P) \leqq \underline{I}_P(V)$, it follows that within this circle $v(P)$ exists, and is approached uniformly by $u_1(P)$ and $u_2(P)$ as ϵ approaches 0. Hence within this circle v is harmonic. By the formation of chains of circles such as those used in the theory of analytic continuation, it may be shown that if V is summable with regard to one point of R, it is summable with regard to all points of R, and that the function $v(P)$ thus defined is harmonic everywhere in R.

Daniell shows that the set of all summable functions is closed with regard to the operations of addition, multiplication by a constant, and taking the absolute value. He also shows that the extended operation I_Q satisfies (C), (A), and (P). Condition (L) is replaced by the important theorem which reads in our terminology as follows: if $\{U_n\}$ is a sequence of summable boundary functions such that $\lim U_n = U$, and if there is a summable function V such that over all C we have $|U_n| \leqq V$ for all n, then U is summable, and $u(P) = \lim u_n(P)$. As one consequence of these theorems, functions belonging to T_0 and T_1 are summable, and all the definitions given of $I_Q(U)$ agree when more than one of them is applicable.

III. The behavior of $I_Q(U)$ in the neighborhood of the boundary

If Q is a point of C, and K is a circle with Q as center, then any function $U(P)$ which is defined and summable on C, bounded, and 0 over that part of C

within K, determines a harmonic function $u(P)$ which approaches 0 as P approaches Q. To prove this, let us take a as the radius of K, and let K' be a concentric circle with radius $a/2$. Let the upper bound of $|U|$ be M. Then we shall define the function $V(P)$ as M outside of K, 0 within K', and $M((2r/a)-1)$ at those points in the annulus between K and K' at a distance r from Q. Clearly V is continuous. The boundary values V on C determine a function $v(P)$ continuous on $R+C$, harmonic on R, and reducing to V on C. By definition

$$|U(P)| \leqq V(P),$$

over C. Hence by a theorem of Daniell

$$|u(P)| \leqq I_P(|U|) \leqq v(P).$$

This holds over all R. Since

$$\lim_{P \to Q} v(P) = 0 = V(Q),$$

we have

$$\lim_{P \to Q} u(P) = 0 = U(Q).$$

This result is capable of immediate generalization. In the first place, by the addition of a constant to U, we get the result that if $W(P)$ is a bounded function summable on C and constant over K,

$$(6) \qquad\qquad \lim_{P \to Q} w(P) = W(Q).$$

Next let W_1 be any bounded summable function defined over C, and let A and B be respectively the upper and lower bounds of the oscillation of W_1 in the neighborhood of the point Q of C. Then given any positive quantity ε, it is possible to describe a circle K about Q as center within which $A + \varepsilon > W_1(P) > B - \varepsilon$. It then follows from the extension of (P) to all summable functions and from (6) that if the functions formed from W_1 by substituting $A + \varepsilon$ or $B - \varepsilon$ within K are summable, the oscillation of $w_1(P)$ in the neighborhood of Q is not greater than from $A + \varepsilon$ to $B - \varepsilon$, and hence, since ε is arbitrary, is not greater than from A to B. If in particular W_1 is continuous at Q, then the function which is W_1 on C and w_1 on R is likewise continuous there.

Daniell has a theorem to the effect that a function equal for every argument to the greater or to the less of two summable functions is itself summable. The function which is the constant H over the part of C within K and 0 elsewhere may be shown without difficulty to belong to T_1, and hence to be summable. Hence the function obtained from W_1 by replacing its values within K by $A + \varepsilon$ or $B - \varepsilon$ is in fact summable.

I have so far been unable to eliminate without any further restriction the condition that W_1 be bounded, although I suspect that this condition is superfluous. The results obtained in this section are valid, of course, if for the word "circle" is substituted "n-sphere", in a space of n dimensions.

IV. CERTAIN SUMMABLE FUNCTIONS

A surface M in n-space is said to have capacity c if there is a positive function $f(P)$ defined over M such that

$$1 = \left(\int_M \right)^{n-1} f(P) \, (PQ)^{n-2} \, dS$$

(in the case $n = 2$,

$$1 = - \int_M f(P) \log PQ \, dS)$$

independently of Q so long as Q remains on M, while

$$\left(\int_M \right)^{n-1} f(P) \, dS = c.$$

A set of points on C is said to have zero measure if the function which is 1 over the set and 0 over the rest of C is summable, and determines a harmonic function 0 over R. I say that if we have given a closed set of points S on the boundary C of an n-dimensional region R for which the Dirichlet problem is solvable, and if S can be included in the interior of a surface M of arbitrarily small capacity, then S is of zero measure.

To prove this, let us remark that we can distribute over the part of C within M a boundary potential that is continuous, 1 on S, and 0 on the boundary of M. This boundary potential, together with the boundary potential 0 over the rest of C, will determine a harmonic function $u(Q)$ over R, which will be uniformly less, within R and outside M, than

(7)
$$\left(\int_M \right)^{n-1} f(P) \, (PQ)^{n-2} \, dS,$$

or its two-dimensional analogue.* Now let Q be a point of R at a distance a from the nearest point of C. Then (7) and hence $u(Q)$ will not exceed $c\,a^{n-2}$ or $-c\log a$, as the case may be. Since C may be made arbitrarily small, the I_Q of our boundary function that is 0 except on S, where it is 1, cannot exceed 0. It follows at once that S is of zero measure. Thus a finite set of points, or, more generally, an $(n-2)$-spread on the $(n-1)$-spread bounding a region in the space of n dimensions is of zero measure.

A function which is bounded and has discontinuities only at a closed set of points of zero measure is summable. Let S be this set. Let M_a be the set of all points on C within a distance of less than a from some point of S. Let $U(P)$ be any function bounded on C and with all its discontinuities on S. It is then possible to form a function $U_a(P)$ continuous over the whole of C and differing from $U(P)$ only over M_a.† It is even possible to keep the set of functions $\{U_a\}$ uniformly bounded. These functions U_a remain summable even if any finite change is made in them over S, since any function finite over S and zero over the rest of C is less in absolute value than a function with Daniell integral everywhere zero, and hence is itself of zero Daniell integral. As a approaches 0, these functions U_a thus modified can be made to approach U boundedly over all C. Hence by a theorem of Daniell which we have already quoted, U_a is summable.

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* In the two-dimensional case, to avoid difficulties due to the change of sign of the logarithm, the unit of measurement should be chosen greater than the greatest linear dimension of C.

† Given any set of points, it is possible to surround it within any assigned distance by a polyhedral boundary. It is hence possible to surround the set by a sequence of such boundaries, each including the next, approaching to the set within any assigned distance. Given any function uniformly continuous on the original set, it is possible to approximate to it by continuous functions on the polyhedral boundaries which all lie between the bounds of the original function, and which all constitute a function continuous over the set of points consisting of the original set, together with the polyhedra. The continuous functions thus obtained may be taken as the boundary values of an infinite set of harmonic functions defined between the successive polyhedra and outside the outermost one. Together these constitute a continuous function extending the original continuous function through the whole of space, and a fortiori over any surface in space.

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